Optimal MM^* bounds for convex bodies.

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Abstract

Let $K \subset \mathbb{R}^n$ be a convex body in isotropic position. We prove the optimal mean-width estimate

$$M^*(K) \le C\sqrt{n \log n}.$$

The main new ingredient is a geometric inequality relating the Gaussian mean of the support function to its mean under the uniform measure on K, obtained through a heat-flow argument. Combined with the Gaussian-log-concave comparison of Eldan and Lehec and the newly available dimension-free bound on the third-moment parameter κ_n, this yields the result. The boundedness of κ_n also makes the mean-norm estimate of Bizeul and Klartag sharp. Combining both estimates yields

$$M(K)M^*(K) \le C \log n,$$

extending Pisier's MM^* estimate to non-symmetric convex bodies and to the isotropic position.

1 Introduction

Throughout the paper, we work in $\mathbb{R}^n$ with $n \ge 2$. A convex body $K \subset \mathbb{R}^n$ is a compact convex subset with non-empty interior. We shall denote by X_K the random vector uniformly distributed in K. More generally, we say that a random vector X is log-concave if it has a density of the form e^{-V}, with $V : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ a convex function. We say that a random vector X is isotropic if it is centered and its covariance matrix is the identity. A convex body K will be called isotropic if the random vector X_K is. Namely,

$$\int_K x \, \mathrm{d}x = 0, \qquad \frac{1}{|K|}\int_K x \otimes x \, \mathrm{d}x = \mathrm{Id},$$

where $|\cdot|$ denotes Lebesgue volume. Note that this convention, which we retain throughout the paper, is sometimes called probabilistic isotropy and is slightly different from volume isotropy, which requires the volume of K to be one and its covariance matrix to be scalar. Any convex body can be made isotropic after applying a suitable invertible affine transformation. In the terminology of convex geometry, an invertible affine image of a convex body is sometimes called a "position".

The support function of K is defined by

$$h_K(x) = \sup_{y \in K} \langle x, y \rangle, \qquad x \in \mathbb{R}^n.$$

If K contains the origin in its interior, we define its gauge by

$$\|x\|_K = \inf\left\{t > 0 :\ \frac{x}{t} \in K\right\}.$$

Under the same assumption, the polar body of K is defined as the body whose gauge is h_K, namely

$$K^\circ = \left\{x \in \mathbb{R}^n :\ \sup_{y\in K}\langle x, y\rangle \le 1\right\}.$$

Note that $\|\cdot\|_K$ and h_K define genuine norms only when K is origin-symmetric.

The mean-norm and mean-width are two fundamental geometric parameters associated with a convex body K, defined respectively by

$$M(K) = \mathbb{E}\,\|\Theta\|_K\,, \qquad M^*(K) = \mathbb{E}h_K(\Theta),$$

where Θ is distributed uniformly on the sphere $\mathbb{S}^{n-1}$. These two quantities are dual to one another since, tautologically,

$$M(K^\circ) = M^*(K).$$

It is not too difficult to provide lower bounds for M and M^*. Indeed, for a fixed volume, Urysohn's inequality states that the mean-width is minimized by the Euclidean ball:

$$M^*(K) \ge \left(\frac{|K|}{|B_2^n|}\right)^{1/n}. \tag{1}$$

Here and below, B_p^n denotes the unit ball of ℓ_p^n. Furthermore, when K contains the origin in its interior, expressing its volume in polar coordinates and using Jensen's inequality gives the corresponding estimate

$$M(K) \ge \left(\frac{|B_2^n|}{|K|}\right)^{1/n}. \tag{2}$$

In particular, the scale-invariant quantity

$$\ell(K) := M(K)M^*(K)$$

satisfies

$$\ell(K) \ge 1.$$

The question of upper-bounding these quantities in an appropriate position is central to convex geometry.

The main result of this paper is the following optimal estimate on the mean-width in isotropic position.

Theorem 1.1. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then*

$$M^*(K) \le C\sqrt{n\log n}, \tag{3}$$

where $C > 0$ is a universal constant.

Theorem 1.1 is optimal, up to the value of the constant C, as follows from the example of the cross-polytope. The best previously known estimate was obtained by Emanuel Milman [32]

$$M^*(K) \lesssim \sqrt{n}\,\log^2 n.$$

Previous results were obtained in [29, 35], which, as observed in [10], could be derived from Paouris's work [33].

The proof of Theorem 1.1 combines two ingredients. The first is the following optimal comparison between Gaussian and log-concave vectors in the gauge order.

Theorem 1.2. *Let X be an isotropic log-concave random vector in $\mathbb{R}^n$, and let $\|\cdot\|$ be a gauge. Then*

$$\frac{\mathbb{E}\,\|G\|}{C\sqrt{\log n}} \leq \mathbb{E}\,\|X\| \leq C\sqrt{\log n}\,\mathbb{E}\,\|G\|\,, \tag{4}$$

where $G \sim N(0, \mathrm{Id})$ is a standard Gaussian vector and $C > 0$ is a universal constant.

The left-hand side of this inequality was established by Bizeul and Klartag [7], while the right-hand side was proved by Eldan and Lehec [15]. In both cases, the original statements involved an additional parameter κ_n, related to the third-order moment tensor of isotropic log-concave vectors. Therefore, the statement given above depends on a dimension-free estimate for κ_n, which is now available [31]. We postpone the discussion of this parameter and of its dimension-free estimate until the end of the introduction.

The second ingredient is the following new inequality.

Theorem 1.3. *Let K be a centered convex body in $\mathbb{R}^n$. Then*

$$\big(\mathbb{E}h_K(G)\big)^2 \leq 2n\,\mathbb{E}h_K(X_K), \tag{5}$$

where $G \sim N(0, \mathrm{Id})$.

Theorem 1.3 does not require an isotropic normalization and admits a self-contained heat-flow proof. It is the main new ingredient in the proof of Theorem 1.1. Taken together, Theorems 1.2 and 1.3 imply Theorem 1.1, as explained in Section 3.

Theorem 1.1 complements the corresponding mean-norm estimate of Bizeul and Klartag, which, combined with the dimension-free bound on κ_n, yields the following.

Theorem 1.4 (Bizeul-Klartag [7])**.** *Let K be an isotropic convex body in $\mathbb{R}^n$. Then*

$$M(K) \leq C\sqrt{\frac{\log n}{n}}, \tag{6}$$

where $C > 0$ is a universal constant.

The bound is sharp, as follows from the example of the cube. Prior to [7], mean-norm estimates in isotropic position were obtained in [18, 19, 38].

Combining both estimates, we obtain

Corollary 1.5. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then*

$$\ell(K) = M(K)M^*(K) \le C \log n. \tag{7}$$

This should be compared with the landmark result of Pisier (see [34]), which states that for every origin-symmetric convex body $K \subset \mathbb{R}^n$, there exists an invertible linear transformation $T \in GL_n$ such that

$$\ell(TK) \le C \log n. \tag{8}$$

Corollary 1.5 extends this estimate to arbitrary convex bodies, without any symmetry assumption, and shows that one may choose the position to be isotropic. Before the present work, MM^* estimates in the non-symmetric setting were pioneered by Rudelson [37] with a bound of order $n^{1/3}$, up to logarithmic factors. More recently, Bizeul and Klartag established an optimal mean-norm bound, up to the value of κ_n, and combined it with Emanuel Milman's mean-width estimate to obtain $\ell(K) \le C \log^3 n$ in isotropic position.

We now include a discussion of the optimality of the results presented above, beyond the isotropic normalization.

Optimality of the estimates. The three estimates

$$M(K) \lesssim \sqrt{\frac{\log n}{n}}, \qquad M^*(K) \lesssim \sqrt{n \log n}, \qquad \ell(K) \lesssim \log n$$

are all optimal in isotropic position. This remains true even if one restricts attention to origin-symmetric convex bodies. As mentioned above, the cube and the cross-polytope saturate the first two inequalities, respectively. The third one is saturated by a product of a cube and a cross-polytope, each factor living in half the dimension. Details are provided in Section 3.3.

For the individual quantities M and M^*, a stronger statement holds. The estimates are in fact affine optimal : they cannot be improved by choosing a different affine position.

Corollary 1.6. *The mean-width and mean-norm estimates are affine optimal. Namely*

$$\sup_K \inf_{T,a} M(TK + a) \simeq \sqrt{\log n}, \qquad \sup_K \inf_{T,a} M^*(TK + a) \simeq \sqrt{\log n}. \tag{9}$$

Here, the suprema run over all convex bodies $K \subset \mathbb{R}^n$ satisfying $|K| = |B_2^n|$, and the infima run over all $T \in SL_n$ and $a \in \mathbb{R}^n$ such that $0 \in \mathrm{int}(TK + a)$. The same conclusions remain valid when the suprema are restricted to origin-symmetric convex bodies.

The situation for the product $\ell(K)$ is subtler. Since $\ell(K)$ is scale invariant, no volume normalization is required. In the class of all convex bodies, one has

Corollary 1.7. *Corollary 1.5 is affine optimal. Namely*

$$\sup_K \inf_{T,a} \ell(TK + a) \simeq \log n, \tag{10}$$

where the supremum runs over all convex bodies $K \subset \mathbb{R}^n$, and the infimum runs over all $T \in GL_n$ and $a \in \mathbb{R}^n$ such that $0 \in \mathrm{int}(TK + a)$.

Thus, Corollary 1.5 is optimal even if one is allowed to choose an arbitrary affine position. In this sense, the extension of Pisier's estimate to non-symmetric convex bodies is optimal. In contrast, if one restricts attention to origin-symmetric convex bodies, it is believed that Pisier's estimate may be improved; see e.g. [3, Section 6.8].

Conjecture 1.8. *For every origin-symmetric convex body $K \subset \mathbb{R}^n$, there exists an invertible linear transformation $T \in GL_n$ such that*

$$\ell(TK) \leq C\sqrt{\log n}, \tag{11}$$

where $C > 0$ is a universal constant.

The conjectural order is the best possible, since it is saturated by the cube, or equivalently by the cross-polytope. Since the logarithmic estimate in Corollary 1.5 is optimal even within the class of origin-symmetric convex bodies, the conjecture above cannot hold in general if one insists on using the isotropic position.

All the optimality statements above, together with the examples establishing them, are discussed in greater detail in Section 3.3.

The parameter κ_n. We conclude the introduction by discussing the parameter κ_n and its newly available dimension-free bound, used in Theorem 1.2. Define

$$\kappa_n = \sup_X \sup_{\theta \in \mathbb{S}^{n-1}} \left\| \mathbb{E}\big[\langle X, \theta \rangle X \otimes X\big] \right\|_{\mathrm{HS}}, \tag{12}$$

where the first supremum runs over all isotropic log-concave random vectors X in $\mathbb{R}^n$, and $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. A dimension-free estimate for κ_n was recently obtained in [31]. For the reader's convenience, we explain how a very short modification of the argument of Chen and Klartag [11] recovers this result. The following sharp thin-shell estimate is proved in [11]

$$\mathrm{Var}(|X|^2) \leq 8n$$

for isotropic log-concave random vectors. A minor anisotropic adaptation of their proof gives

$$\mathrm{Var}(|X|^2) \leq 8\,\mathrm{Tr}(\Sigma^2) \tag{13}$$

for every centered log-concave random vector X with covariance matrix Σ.

Consequently, if X is isotropic and A is symmetric positive semidefinite, then

$$\mathrm{Var}\big(\langle AX, X\rangle\big) \leq 8\,\mathrm{Tr}(A^2).$$

For an arbitrary symmetric matrix, splitting A into its positive and negative parts gives

$$\mathrm{Var}\big(\langle AX, X\rangle\big) \leq 16\,\mathrm{Tr}(A^2).$$

This is sufficient to obtain

$$\kappa_n^2 \leq 16, \tag{14}$$

see Corollary 2.7 below. With a little more work one may obtain the slightly better estimate $\kappa_n^2 \leq 8$, but this is not relevant for our purposes.

The parameter κ_n first appeared in Eldan's original paper on the stochastic-localization process [14]. Let us briefly recall its role. We refer to the lecture notes of Klartag and Lehec [28] for background on the process, its history and applications.

Let $X \sim \mu$ be isotropic and log-concave, let $(B_t)_{t\geq 0}$ be an independent Brownian motion, and let

$$\mu_t = \mathcal{L}\big(X \,\big|\, tX + B_t\big).$$

The stochastic process $(\mu_t)_{t\geq 0}$ is called the stochastic-localization process initiated at μ. Write $A_t = \mathrm{Cov}(\mu_t)$, so that $A_0 = \mathrm{Id}$. Put

$$T_0 = \frac{c}{\kappa_n^2 \log n} \tag{15}$$

for an appropriate constant $c > 0$. With probability at least $1 - Ce^{-c/T_0}$, one has

$$\frac{1}{2}\,\mathrm{Id} \preceq A_t \preceq 2\,\mathrm{Id}, \qquad 0 \leq t \leq T_0; \tag{16}$$

see [7, Proposition 2.3]. The upper bound in (16) is essentially due to Eldan [14], while the lower bound was established by Bizeul and Klartag [7], following previous work of Lee and Vempala [30].

In view of (14), we may take $T_0 = c/\log n$, and, for a suitable choice of $c > 0$, the failure probability in (16) is at most n^{-10}. The order $1/\log n$ is sharp. It is explained in [28] that products of centered exponentials saturate the upper bound, while an analogous argument shows that the uniform distribution on the cube saturates the lower bound.

In [23] Klartag establishes an improved log-concave Lichnerowicz inequality and uses it to prove that

$$\Psi_n^2 \lesssim T_0^{-1/2}, \tag{17}$$

where Ψ_n is the KLS constant [21], that is, the best constant such that for every isotropic log-concave vector X, and every Lipschitz function f, we have the Poincaré inequality

$$\mathrm{Var}\, f(X) \leq \Psi_n^2\, \mathbb{E}|\nabla f(X)|^2.$$

Therefore, using (15), (17) and the dimension-free estimate (14), one obtains

$$\Psi_n \leq C(\log n)^{1/4}, \tag{18}$$

as recorded in [31].

The same covariance estimate underlies the gauge comparisons in [15, 7], which were established with a factor of order $T_0^{-1/2}$. Therefore, using (15) and (14) yields Theorem 1.2.

The paper is organized as follows. Section 2 is purely expository. It provides a sketch of the argument of Chen and Klartag and explains the minor anisotropic adaptation of their proof yielding the sharp generalized variance bound, and its application to κ_n, as recorded in [31]. Section 3 proves Theorem 1.3 and use it to deduce Theorem 1.1. It also contains a discusseion on the optimality of the results, and records applications of the MM^* estimate to Banach-Mazur distances and the flatness constant.

Notation. Throughout the paper, $C, c > 0$ denote universal constants whose values may change from one occurrence to the next. We write $A \lesssim B$ if $A \leq CB$, and $A \simeq B$ if both $A \lesssim B$ and $B \lesssim A$. The notation $|x|$ denotes the Euclidean norm of a vector, whereas $|K|$ denotes the Lebesgue volume of a measurable set. For a matrix A, we write

$$\|A\|_{\mathrm{HS}} = \sqrt{\mathrm{Tr}(A^\top A)},$$

and $A \preceq B$ means that $B - A$ is positive semidefinite. Finally, for $x, y \in \mathbb{R}^n$, we set $x \otimes y = (x_i y_j)_{i,j=1}^n$.

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2 An anisotropic version of the Chen-Klartag argument

We stress that this section is purely expository and contains no original content. It is based entirely on the preprint of Chen and Klartag [11], and describes the minor anisotropic adaptation of their argument, first recorded in [31], which yields a generalized thin-shell estimate.

The thin-shell problem originates in the work of Anttila, Ball and Perissinaki [2] and was subsequently formulated as the variance conjecture by Bobkov and Koldobsky [8]. It asks whether every isotropic log-concave random vector X in $\mathbb{R}^n$ satisfies

$$\mathrm{Var}(|X|^2) \leq Cn.$$

After a long sequence of developments, this conjecture was recently resolved by Klartag and Lehec [25], following a breakthrough of Guan [20]. Most of the preceding approaches used Eldan's stochastic-localization process; see, among others, [14, 30, 26, 23, 20, 25]. Chen and Klartag [11] gave a different proof, based on log-concave moment measures, and obtained the optimal constant 8. The purpose of this section is to explain the minor anisotropic modification of their argument which gives

$$\mathrm{Var}(|X|^2) \leq 8\,\mathrm{Tr}(\Sigma^2) \tag{19}$$

for a centered log-concave vector with covariance matrix Σ, as recorded in [31]. The question of studying thin-shell estimates beyond the isotropic setting was put forward in [1].

A common starting point in recent approaches to thin-shell problems is the H^{-1} inequality of Klartag [22], which was extended to general log-concave measures in [5]. For a centered function $f \in L^2(\mu)$, set

$$\|f\|_{H^{-1}(\mu)} = \sup\left\{ \int fg \,\mathrm{d}\mu : g \in C_c^\infty(\mathbb{R}^n),\ \int |\nabla g|^2 \,\mathrm{d}\mu \leq 1 \right\}. \tag{20}$$

Here and below, $C_c^\infty(\mathbb{R}^n)$ is the class of smooth compactly supported functions. The H^{-1} inequality asserts that if f is locally Lipschitz and $f, \partial_1 f, \ldots, \partial_n f \in L^2(\mu)$ are centered, then

$$\mathrm{Var}_\mu(f) \leq \sum_{i=1}^n \|\partial_i f\|_{H^{-1}(\mu)}^2 . \tag{21}$$

Applying this inequality to $f(x) = |x|^2 - \int |x|^2 \, \mathrm{d}\mu(x)$ gives

$$\mathrm{Var}(|X|^2) \leq 4 \sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2 . \tag{22}$$

The right-hand side admits a simple interpretation in terms of Stein kernels. Recall that a square-integrable matrix field $\tau : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a Stein kernel for μ if

$$\int x \cdot F(x) \, \mathrm{d}\mu(x) = \int \langle \tau(x), DF(x) \rangle_{\mathrm{HS}} \, \mathrm{d}\mu(x) \tag{23}$$

for every $F \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, where DF denotes the Jacobian matrix of F.

Proposition 2.1. *Let τ be a Stein kernel for μ. Then*

$$\mathrm{Var}(|X|^2) \leq 4 \int \|\tau(x)\|_{\mathrm{HS}}^2 \, \mathrm{d}\mu(x). \tag{24}$$

Proof. For $g \in C_c^\infty(\mathbb{R}^n)$, applying the Stein identity to $F = g e_i$ gives

$$\int x_i g \, \mathrm{d}\mu = \int \sum_{j=1}^n \tau_{ij} \partial_j g \, \mathrm{d}\mu.$$

By Cauchy-Schwarz,

$$\|x_i\|_{H^{-1}(\mu)}^2 \leq \int \sum_{j=1}^n \tau_{ij}^2 \, \mathrm{d}\mu.$$

Summing over i and using (22) proves (24). $\square$

In fact, Courtade, Fathi and Pananjady [13] construct a distinguished Stein kernel τ_L, related to the Langevin generator of μ, satisfying

$$\int \|\tau_L\|_{\mathrm{HS}}^2 \, \mathrm{d}\mu = \sum_{i=1}^n \|x_i\|_{H^{-1}(\mu)}^2 . \tag{25}$$

Thus the thin-shell problem is reduced to constructing a Stein kernel with sufficiently small Hilbert-Schmidt energy. The proof of Chen and Klartag uses a different kernel, built from moment measures, which we now describe.

2.1 Moment measures and the Chen-Klartag argument

Let μ be a centered log-concave probability measure on $\mathbb{R}^n$, with covariance matrix

$$\Sigma = \int x \otimes x \, \mathrm{d}\mu(x).$$

The moment-measure theorem of Cordero-Erausquin and Klartag [12] provides an essentially continuous convex function $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, unique up to translation of its argument, such that

$$\int_{\mathbb{R}^n} e^{-\psi(y)} \, \mathrm{d}y = 1, \qquad (\nabla\psi)_{\#}\big(e^{-\psi(y)} \, \mathrm{d}y\big) = \mu. \tag{26}$$

Following Chen and Klartag [11], we assume temporarily that μ has density e^{-V} on a bounded open convex set K, with smooth boundary, and that V is smooth and convex in a neighborhood of K. Under these assumptions, Klartag [24] proved that ψ is smooth and strictly convex, and that $\nabla\psi : \mathbb{R}^n \to K$ is a diffeomorphism. Set

$$\mathrm{d}\nu(y) = e^{-\psi(y)} \, \mathrm{d}y, \qquad H(y) = D^2\psi(y). \tag{27}$$

The Hessian H, transported to the target measure, is a symmetric positive-definite Stein kernel, as observed by Fathi [16, Theorem 2.3]. We denote it by

$$\tau_M(x) = H\big((\nabla\psi)^{-1}(x)\big). \tag{28}$$

Indeed, formally, if $F \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, integration by parts and the push-forward relation give

$$\begin{aligned}
\int_K x \cdot F(x) \, \mathrm{d}\mu(x) &= \int_{\mathbb{R}^n} \nabla\psi(y) \cdot F(\nabla\psi(y)) \, \mathrm{d}\nu(y) \\
&= \int_{\mathbb{R}^n} \mathrm{Tr}\big(H(y) DF(\nabla\psi(y))\big) \, \mathrm{d}\nu(y) \\
&= \int_K \langle \tau_M(x), DF(x) \rangle_{\mathrm{HS}} \, \mathrm{d}\mu(x).
\end{aligned}$$

Moreover,

$$\int_{\mathbb{R}^n} H \, \mathrm{d}\nu = \int_K x \otimes x \, \mathrm{d}\mu(x) = \Sigma. \tag{29}$$

The change-of-variables formula in (26) yields the Monge-Ampère equation

$$e^{-V(\nabla\psi)} \det H = e^{-\psi}. \tag{30}$$

As explained in [24], it is useful to regard H as a Riemannian metric on $\mathbb{R}^n$, endowed with the probability measure $\nu = e^{-\psi} \, \mathrm{d}y$. The associated symmetric generator is

$$\mathcal{L}f = e^{\psi} \operatorname{div}\big(e^{-\psi} H^{-1} \nabla f\big), \tag{31}$$

and its carré du champ is

$$\Gamma(f,g) = \left\langle H^{-1}\nabla f, \nabla g \right\rangle. \tag{32}$$

Thus

$$\int g\,\mathcal{L}f\,\mathrm{d}\nu = -\int \Gamma(f,g)\,\mathrm{d}\nu.$$

The Brascamp-Lieb inequality [9] for the measure ν reads

$$\mathrm{Var}_\nu(f) \le \int \Gamma(f)\,\mathrm{d}\nu, \tag{33}$$

where $\Gamma(f) = \Gamma(f,f)$. When $\mathcal{L}$ is applied to a matrix field, it acts entrywise.

We now give a brief overview of the part of the Chen-Klartag argument used below. Write

$$\psi_i = \partial_i\psi, \qquad \psi_{ij} = \partial_i\partial_j\psi, \qquad \psi_{ijk} = \partial_i\partial_j\partial_k\psi.$$

For $i = 1,\dots,n$, put

$$C_i = \partial_i H, \qquad (C_i)_{jk} = \psi_{ijk}.$$

Differentiating the Monge-Ampère equation produces the nonnegative matrix fields

$$A = H\big(D^2V\circ\nabla\psi\big)H \succeq 0, \qquad Q_{ij} = \mathrm{Tr}(H^{-1}C_iH^{-1}C_j), \quad Q \succeq 0, \tag{34}$$

and the identity

$$\mathcal{L}H + H = A + Q. \tag{35}$$

This is Lemma 3.1 of [11]. It refines an earlier argument of Klartag [24] who established the positivity of the left-hand side of (35).

We are interested in bounding the Hilbert-Schmidt norm of the kernel τ_M, namely

$$N_2 = \int_{\mathbb{R}^n} \mathrm{Tr}(H^2)\,\mathrm{d}\nu = \int \|\tau_M(x)\|_{\mathrm{HS}}^2\,\mathrm{d}\mu(x). \tag{36}$$

At a fixed point, choose coordinates in which

$$H = \mathrm{diag}(\lambda_1,\dots,\lambda_n), \qquad \lambda_i > 0,$$

and define

$$\begin{aligned}
d_2 &= \sum_{a,i,j=1}^n \frac{\psi_{aij}^2}{\lambda_a},\\
a_2 &= \mathrm{Tr}(HA),\\
q_2 &= \mathrm{Tr}(HQ) = \sum_{a,i,j=1}^n \frac{\lambda_a}{\lambda_i\lambda_j}\psi_{aij}^2.
\end{aligned}$$

Put

$$D_2 = \int d_2\,\mathrm{d}\nu, \qquad A_2 = \int a_2\,\mathrm{d}\nu, \qquad Q_2 = \int q_2\,\mathrm{d}\nu. \tag{37}$$

The two identities used below are Lemmas 3.3 and 3.5 of [11].

Lemma 2.2 (Chen-Klartag)**.** *Under the preceding regularity assumptions,*

$$N_2 = D_2 + A_2 + Q_2, \tag{38}$$

and

$$q_2 - d_2 = \frac{1}{6} \sum_{a,i,j=1}^{n} \frac{\psi_{aij}^2}{\lambda_a \lambda_i \lambda_j} \Big[(\lambda_a - \lambda_i)^2 + (\lambda_i - \lambda_j)^2 + (\lambda_j - \lambda_a)^2 \Big] \geq 0. \tag{39}$$

In particular, $Q_2 \geq D_2$*.*

The identity (38) follows by applying the diffusion product rule to $\mathrm{Tr}(H^2)$ and integrating (35). The inequality (39) relies on the symmetry of the third derivative tensor $D^3\psi$. We may now state the Chen-Klartag bootstrap [11, Theorem 1.5].

Theorem 2.3 (Chen-Klartag)**.** *If* μ *is isotropic, then*

$$\int \|\tau_M(x)\|_{\mathrm{HS}}^2 \, \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \mathrm{Tr}(H^2) \, \mathrm{d}\nu \leq 2n. \tag{40}$$

Proof. If μ is isotropic, then (29) gives $\int H \, \mathrm{d}\nu = \mathrm{Id}$. Applying (33) to every entry of H yields

$$N_2 - n = \int \|H - \mathrm{Id}\|_{\mathrm{HS}}^2 \, \mathrm{d}\nu \leq D_2. \tag{41}$$

On the other hand, Lemma 2.2 gives

$$N_2 = D_2 + A_2 + Q_2 \geq 2D_2.$$

Combining the two inequalities gives $N_2 \leq 2n$. □

2.2 The anisotropic adaptation

The preceding proof does not use isotropy until the Brascamp-Lieb step (41). Replacing that single line gives the following anisotropic version.

Proposition 2.4. *If* μ *has covariance matrix* Σ*,*

$$\int \|\tau_M(x)\|_{\mathrm{HS}}^2 \, \mathrm{d}\mu(x) = \int_{\mathbb{R}^n} \mathrm{Tr}(H^2) \, \mathrm{d}\nu \leq 2 \, \mathrm{Tr}(\Sigma^2). \tag{42}$$

Proof. By (29), $\int H \, \mathrm{d}\nu = \Sigma$. Applying the Brascamp-Lieb inequality (33) to every entry of H and summing gives

$$\begin{aligned} N_2 - \mathrm{Tr}(\Sigma^2) &= \int \|H - \Sigma\|_{\mathrm{HS}}^2 \, \mathrm{d}\nu \\ &= \sum_{i,j=1}^{n} \mathrm{Var}_\nu(H_{ij}) \leq \sum_{i,j=1}^{n} \int \Gamma(H_{ij}) \, \mathrm{d}\nu = D_2. \end{aligned} \tag{43}$$

Since $A_2 \geq 0$ and $Q_2 \geq D_2$, we still have

$$N_2 = D_2 + A_2 + Q_2 \geq 2D_2.$$

Using (43), we conclude that

$$N_2 \geq 2\big(N_2 - \mathrm{Tr}(\Sigma^2)\big),$$

which is equivalent to (42). $\square$

Therefore, we arrive at the following sharp anisotropic thin-shell estimate, recorded in [31].

Theorem 2.5. *Let X be a centered log-concave random vector in $\mathbb{R}^n$ with covariance matrix Σ. Then*

$$\mathrm{Var}(|X|^2) \leq 8\,\mathrm{Tr}(\Sigma^2). \tag{44}$$

The constant 8 is optimal.

Proof. Under the regularity assumptions, Proposition 2.1 applied to the Stein kernel τ_M and Proposition 2.4 give

$$\mathrm{Var}(|X|^2) \leq 4 \int \|\tau_M\|_{\mathrm{HS}}^2 \,\mathrm{d}\mu \leq 8\,\mathrm{Tr}(\Sigma^2).$$

As explained in [11, Section 3], the regularity assumptions may be removed by approximation. Sharpness follows by considering independent centered exponential variables with the appropriate variances in a basis diagonalizing Σ. $\square$

We next pass from (44) to quadratic forms.

Corollary 2.6. *Let X be an isotropic log-concave random vector in $\mathbb{R}^n$. If $A \succeq 0$ is symmetric, then*

$$\mathrm{Var}\big(\langle AX, X\rangle\big) \leq 8\,\mathrm{Tr}(A^2). \tag{45}$$

For an arbitrary symmetric matrix A,

$$\mathrm{Var}\big(\langle AX, X\rangle\big) \leq 16\,\mathrm{Tr}(A^2). \tag{46}$$

Proof. For the first statement, apply Theorem 2.5 to $Y = A^{1/2}X$. For a general symmetric matrix, split it into positive and negative parts $A = A_+ - A_-$, where $A_+, A_- \succeq 0$ and $A_+A_- = 0$ and use (45).

$$\begin{aligned}\mathrm{Var}\big(\langle AX, X\rangle\big) &\leq 16\,\mathrm{Tr}(A_+^2) + 16\,\mathrm{Tr}(A_-^2)\\ &= 16\,\mathrm{Tr}(A^2).\end{aligned}$$

$\square$

We now return to the third-moment parameter κ_n defined in (12).

Corollary 2.7. *One has*

$$\kappa_n^2 \leq 16. \tag{47}$$

Proof. Let X be isotropic and log-concave, let $\theta \in \mathbb{S}^{n-1}$, and set

$$B = \mathbb{E}\big[\langle X, \theta\rangle X \otimes X\big].$$

Then B is symmetric and

$$\begin{aligned}\|B\|_{\mathrm{HS}}^2 &= \mathbb{E}\left(\langle X, \theta\rangle . \langle BX, X\rangle\right)\\ &\leq \mathrm{Var}\big(\langle BX, X\rangle\big)^{1/2} \leq 4\,\|B\|_{\mathrm{HS}}\,,\end{aligned}$$

where we used isotropy and (46). Hence $\|B\|_{\mathrm{HS}} \leq 4$, and the result follows. □

3 Mean-width bound and applications

In this section we prove the main result of the paper. We first prove the geometric inequality of Theorem 1.3 using heat flow and combine it with the gauge-order comparison of Theorem 1.2 to deduce Theorem 1.1. We then establish the isotropic and affine optimality statements from the introduction, before recording the consequences for Banach-Mazur distances and the flatness constant.

Before anything, we recall the very standard fact that spherical integration may be replaced by Gaussian integrations in the definition of the mean-norm and mean-width. Indeed if G is a standard Gaussian vector in $\mathbb{R}^n$, then $G = |G|\Theta$, where Θ is uniformly distributed on the sphere, $|G|$ is the Euclidean norm of G and both quantities are independent. Therefore, by homogeneity, for any K,

$$\mathbb{E}\,\|G\|_K = \mathbb{E}|G|\,M(K) \simeq \sqrt{n}M(K), \qquad \mathbb{E}h_K(G) = \mathbb{E}|G|\,M^*(K) \simeq \sqrt{n}M^*(K), \tag{48}$$

where we used that $\mathbb{E}|G| \simeq \sqrt{n}$.

3.1 A heat-flow argument

The main new ingredient for the mean-width estimate is Theorem 1.3, whose proof we now give.

Proof of Theorem 1.3. By a standard approximation argument, we may assume that K has a smooth boundary. Let X_K be uniformly distributed in the centered convex body K, and write

$$W_K = \mathbb{E}h_K(X_K), \qquad W_G = \mathbb{E}h_K(G).$$

Let $(B_s)_{s\geq 0}$ be a standard Brownian motion, independent of X_K, and set

$$F(s) = \mathbb{E}h_K(X_K + B_s) = \frac{1}{|K|}\int_K P_s h_K(x)\,\mathrm{d}x,$$

where P_s is the heat semigroup with generator $\Delta/2$. By the divergence theorem,

$$F'(s) = \frac{1}{2|K|}\int_{\partial K} \langle \nabla P_s h_K(x), n_K(x)\rangle\;\mathrm{d}S(x),$$

where n_K denotes the outer unit normal and $\mathrm{d}S$ the surface measure. The support function is differentiable almost everywhere and $\nabla h_K(y) \in K$ at every differentiability point. Hence, by convexity of K, the average

$$\nabla P_s h_K = P_s(\nabla h_K)$$

also takes values in K. For $x \in \partial K$,

$$\langle \nabla P_s h_K(x), n_K(x) \rangle \le h_K(n_K(x)) = \langle x, n_K(x) \rangle .$$

A second application of the divergence theorem gives

$$F'(s) \le \frac{1}{2|K|} \int_{\partial K} \langle x, n_K(x) \rangle \, \mathrm{d}S(x) = \frac{n}{2}.$$

Consequently,

$$F(s) \le W_K + \frac{ns}{2}.$$

Choosing $s_* = 2W_K/n$, we find that $F(s_*) \le 2W_K$. On the other hand, since X_K is centered, Jensen's inequality yields

$$F(s_*) = \mathbb{E} h_K(X_K + \sqrt{s_*} G) \ge \mathbb{E} h_K(\sqrt{s_*} G) = \sqrt{s_*}\, W_G.$$

Therefore, combining both inequalities yields

$$W_G^2 \le \frac{4W_K^2}{s_*} = 2nW_K,$$

and the proof is complete. □

3.2 Proof of Theorem 1.1

We now deduce Theorem 1.1. Let K be isotropic. Since $h_K = \|\cdot\|_{K^\circ}$ is a gauge, Theorem 1.2 gives

$$\mathbb{E} h_K(X_K) \le C\sqrt{\log n}\, \mathbb{E} h_K(G).$$

Combining this estimate with Theorem 1.3, we obtain

$$\big(\mathbb{E} h_K(G)\big)^2 \le Cn\sqrt{\log n}\, \mathbb{E} h_K(G),$$

and hence

$$\mathbb{E} h_K(G) \le Cn\sqrt{\log n}.$$

Using (48), we conclude that

$$M^*(K) \le C\sqrt{n \log n},$$

as claimed.

3.3 Optimality of the estimates

We now prove the optimality statements made in the introduction. We first consider the isotropic position and then pass to arbitrary affine positions.

Sharpness in isotropic position. Consider the convex bodies

$$Q_n = \sqrt{3}\, B_\infty^n, \qquad P_n = \sqrt{\frac{(n+1)(n+2)}{2}}\, B_1^n.$$

The bodies Q_n and P_n are respectively the isotropic cube and the isotropic cross-polytope. Since $\mathbb{E}\,\|G\|_\infty \simeq \sqrt{\log n}$, formula (48) gives

$$M(Q_n) \simeq \sqrt{\frac{\log n}{n}}, \qquad M^*(P_n) \simeq \sqrt{n \log n}. \tag{49}$$

Thus the mean-norm and mean-width estimates are separately sharp in isotropic position, even among origin-symmetric bodies.

The product estimate is also sharp in the symmetric isotropic class. For simplicity, let the ambient dimension be $2m$ and consider

$$K = Q_m \times P_m.$$

The body K is origin-symmetric and isotropic. Let $G = (G_1, G_2)$, where G_1, G_2 are independent standard Gaussian vectors in $\mathbb{R}^m$. We have

$$\|(x,y)\|_K = \max\{\|x\|_{Q_m}, \|y\|_{P_m}\}, \qquad h_K(x,y) = h_{Q_m}(x) + h_{P_m}(y).$$

Moreover,

$$\mathbb{E}\,\|G_1\|_{Q_m} \simeq \sqrt{\log m}, \qquad \mathbb{E}\,\|G_2\|_{P_m} \simeq 1,$$

and

$$\mathbb{E} h_{Q_m}(G_1) \simeq m, \qquad \mathbb{E} h_{P_m}(G_2) \simeq m\sqrt{\log m}.$$

Therefore,

$$M(K) \simeq \sqrt{\frac{\log m}{m}}, \qquad M^*(K) \simeq \sqrt{m \log m},$$

and hence

$$\ell(K) \simeq \log m. \tag{50}$$

This proves the sharpness of the isotropic MM^* estimate, even in the origin-symmetric class.

This example is nevertheless compatible with Conjecture 1.8. Indeed, for the linear image

$$\widetilde{K} = Q_m \times \frac{1}{\sqrt{\log m}} P_m,$$

the same computation gives

$$M(\widetilde{K}) \simeq \sqrt{\frac{\log m}{m}}, \qquad M^*(\widetilde{K}) \simeq \sqrt{m}, \qquad \ell(\widetilde{K}) \simeq \sqrt{\log m}.$$

Minimal mean-width position. We shall use a characterization of the minimal mean-width position due to Giannopoulos and Milman [17]. Let σ denote the rotationally invariant probability measure on $\mathbb{S}^{n-1}$. We say that a convex body K is in minimal mean-width position if

$$M^*(K) \le M^*(TK) \qquad \text{for every } T \in SL_n.$$

Giannopoulos and Milman proved that this is equivalent to the measure $h_K\,\mathrm{d}\sigma$ having a scalar covariance matrix,

$$\int_{\mathbb{S}^{n-1}} h_K(u)\, u \otimes u \,\mathrm{d}\sigma(u) = \frac{M^*(K)}{n}\,\mathrm{Id}\,. \tag{51}$$

If K is 1-symmetric in some basis, that is, invariant under coordinate permutations and sign changes, so is the measure $h_K\,\mathrm{d}\sigma$, therefore its covariance matrix is scalar and K is in minimal mean-width position.

The same conclusion applies to the mean-norm M : If K is 1-symmetric then it is in minimal mean-norm position. Indeed, by what precedes, its polar K°, which inherits the same symmetries, has minimal mean width. Now for any $T \in SL_n$,

$$M(TK) = M^*((TK)^\circ) = M^*((T^{-1})^T K^\circ) \ge M^*(K^\circ) = M(K).$$

It follows that the standard positions of the cube and the cross-polytope simultaneously minimize M and M^* over volume-preserving linear images.

When working in the symmetric class, it is enough to consider linear images of K, rather than affine ones. Indeed, while the mean-width is invariant by translation, the mean-norm can only increase if an origin-symmetric body is translated. For completeness, we give a short proof of this elementary fact.

For a convex body L containing the origin, write

$$\rho_L(u) = \max\{t \ge 0 : \; tu \in L\},$$

so that $\|u\|_L = 1/\rho_L(u)$. Let $K = -K$ and assume that $0 \in \mathrm{int}(K+b)$. Put $r_+ = \rho_{K+b}(u)$ and $r_- = \rho_{K+b}(-u)$. Then $r_+u - b \in K$, while symmetry gives $r_-u + b \in K$. By convexity,

$$\frac{r_+ + r_-}{2} u \in K,$$

and therefore $r_+ + r_- \le 2\rho_K(u)$. Hence

$$\frac{1}{2}\left(\|u\|_{K+b} + \|-u\|_{K+b}\right) = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right) \ge \frac{2}{r_+ + r_-} \ge \|u\|_K\,.$$

Integrating over the sphere yields

$$M(K+b) \ge M(K). \tag{52}$$

Affine optimality of M and M^*. We now prove Corollary 1.6. Let K be a convex body with $|K| = |B_2^n|$ and let $\widetilde{K} = TK + a$, $T \in SL_n$ be an affine image of K such that $\widetilde{K}$ is centered and

$$\mathrm{Cov}(\widetilde{K}) = \int_{\widetilde{K}} x \otimes x \, \mathrm{d}x = \lambda_K^{-2} I_n$$

for some $\lambda_K > 0$. The solution of the slicing problem [27, 6] says exactly that

$$\lambda_K \simeq \sqrt{n}.$$

Applying Theorems 1.4 and 1.1 to the isotropic convex body $\lambda_K \widetilde{K}$, we obtain

$$M(\widetilde{K}) \lesssim \sqrt{\log n}, \qquad M^*(\widetilde{K}) \lesssim \sqrt{\log n}.$$

This proves the upper bounds in (9).

For the reverse direction, set $\widetilde{P}_n$ and $\widetilde{Q}_n$ to be scaled versions of Q_n and P_n so that their volumes are equal to $|B_2^n|$. The scaling factors are of order $n^{-1/2}$, so (49) becomes

$$M(\widetilde{Q}_n) \simeq \sqrt{\log n}, \qquad M^*(\widetilde{P}_n) \simeq \sqrt{\log n}.$$

By the preceding paragraph, $\widetilde{Q}_n$ and $\widetilde{P}_n$ are in minimal mean-norm and mean-width position. This proves the corollary, including its restriction to origin-symmetric convex bodies.

The affine MM^* problem. We finally prove Corollary 1.7. The upper bound in (10) follows by placing an arbitrary convex body in isotropic position and applying Corollary 1.5.

The reverse direction was settled by Banaszczyk, Litvak, Pajor and Szarek [4], who proved that for any simplex Δ containing the origin in its interior,

$$\ell(\Delta) \gtrsim \log n.$$

Together with the upper bound, this proves (10).

Finally, the conjectural $\sqrt{\log n}$ order in the origin-symmetric class is itself optimal. Indeed, as previously discussed, the standard position of the cube B_∞^n (or equivalently, the cross polytope) simultaneously minimizes M and M^* over SL_n. Since ℓ is scale invariant, this implies

$$\inf_{T \in GL_n} \ell(TB_\infty^n) = \ell(B_\infty^n) \simeq \sqrt{\log n}.$$

On the other hand, (50) shows that the isotropic position may have an MM^* product of order $\log n$, even in the origin-symmetric class. Therefore, a proof of Conjecture 1.8 should not use the isotropic position.

3.4 Applications of the MM* estimate

We conclude with two consequences of the bounds on M and M^*. Recall that the Banach-Mazur distance between two convex bodies $K, L \subset \mathbb{R}^n$ is defined by

$$d_{BM}(K, L) = \inf \left\{ \lambda \geq 1 : \ \exists T \in GL_n, \ x, y \in \mathbb{R}^n, \ TK + x \subset L + y \subset \lambda(TK + x) \right\}.$$

The argument of [7] gives, after putting $K, L \subset \mathbb{R}^n$ in isotropic position,

$$d_{BM}(K,L) \le C\Big(n \max\{M(K), M(L)\} + \max\{M^*(K), M^*(L)\}\Big)^2$$

for all $n \ge 2$. Together with (3) and (6), this yields

$$d_{BM}(K,L) \le Cn \log n. \tag{53}$$

The relevance of MM^* bounds to Banach-Mazur distances was pioneered by Rudelson [37].

The same estimates also improve the flatness bound. Recall that the flatness constant $\mathrm{Flt}(n)$ is the smallest F such that every convex body $K \subset \mathbb{R}^n$ satisfying $K \cap \mathbb{Z}^n = \varnothing$ has lattice width at most F, that is,

$$\min_{z \in \mathbb{Z}^n \setminus \{0\}} \left(\sup_{x \in K} \langle x, z \rangle - \inf_{x \in K} \langle x, z \rangle \right) \le F.$$

It follows from the MM^* estimate and the argument of Banaszczyk, Litvak, Pajor and Szarek [4] that

$$\mathrm{Flt}(n) \le Cn \log n. \tag{54}$$

This improves the bound $O(n \log^2 n)$ obtained by Reis and Rothvoss [36].

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